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Estimation of micropolar elastic moduli by inversion of vibrational data

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Abstract

A nonlinear wave theory is used to construct an inverse approach for estimation the micropolar elastic moduli of a micropolar plate. The analytical formulation given by this theory is incorporated in a nonlinear minimization algorithm, which minimize the least square “distance” between computed and measured natural frequencies. The minimization algorithm has two steps: a genetic algorithm and a global minimisation procedure (quasi-Newton). Based on the interrelations between natural frequencies and the elastic properties of the material we show that it is possible to reconstruct in a unique manner the unknown micropolar elastic moduli.

1. Introduction

The classical elasticity theory is believed to be inadequate for describing the dynamics of a plate consisted from a micropolar material. A homogeneous isotropic micropolar material is a material characterized by a continuum in which rigid grains of infinitesimal size are uniformly distributed in an elastic matrix. Homogeneity and isotropy are macro-properties of the medium. The effect of granular structure becomes important in transmitting waves of small wavelength and high frequency. When the wavelength is comparable with the average grain size, the motion of the grains must be taken into account. A complete derivation of the micropolar elasticity equations was given in 1964 by Eringen and Suhubi [1], Eringen, 1966 [2], Sung Kim and Eringen, 1973 [3], Gauthier, 1982 [4]. This work suggests a method to identify the micropolar elastic moduli on the basis of the vibrating data, for a plate made from a micropolar material fabricated by Gauthier [4] in which uniformly distributed rigid aluminium shot was cast in an elastic epoxy matrix. The effect of shear waves is taken into consideration by considering a third-order theory for the displacement field. There are four basic waves travelling at four distinct phase velocities into the plate. But only two coupled shear waves having the wavelengths comparable to the size of grains are used to estimate the micropolar elastic moduli. Mathematically, an inverse problem is ill-posed and has to be overcome by developments of new computational methods, introduction of new objective function into the optimal control algorithm, new experimental procedures, etc (Frederiksen,

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1992 [4], Fallstrom and. Jonsson, 1991 [5], Chiroiu et al., 1997 [6], Badea et al., 2000, [7]}. Usually, simple inversions are performed using some methods as genetic algorithms, the classical iterative ART-type method (Algebraic reconstruction technique), or neural networks methods (derived through a modelling of human brain).

In this paper we use a genetic algorithm (GA) as a first step to obtain the initial guess of the model parameters (Goldberg, 1989 [8], Tanaka and Nakamura, 1994 [9], Chiroiu et al., 1999 [10]). Then, a quasi-Newton optimisation method is used as a second step to find the final solution. The proposed method gives the unknown parameters for a more precise inverse analysis.

2. Basic Equations

Eringen 's theory of micropolar elasticity is based upon the following equations:

Balance of momentum:

$$\sigma_{kl,k} - \rho \ddot{u}_l = 0 \tag{2.1}$$

Balance of moment of momentum

$$m_{rk,r} + e_{klr}\sigma_{lr} - \rho j \ddot{\varphi}_k = 0 \tag{2.2}$$

Conservation of energy

$$\rho\dot{\varepsilon} = \sigma_{kl}(v_{l,k} - e_{klr}\xi_r) + m_{kl}\xi_{l,k} \tag{2.3}$$

Constitutive equations

$$\sigma_{kl} = \lambda u_{r,r}\delta_{kl} + \mu(u_{k,l} + u_{l,k}) + \vartheta(u_{l,k} - e_{klr}\varphi_r) \tag{2.4}$$

$$m_{kl} = \alpha\varphi_{r,r}\delta_{kl} + \beta\varphi_{k,l} + \gamma\varphi_{l,k} \tag{2.5}$$

where $C = \{\lambda,\mu,\vartheta,\alpha,\beta,\rho\}$ are the elastic moduli (λ,μ the Lame elastic constants, ϑ,α,β the additional constants in micropolar theory and ρ the density of the material, σ_{kl} is stress tensor, ρ is density, u_k is displacement vector, ε is internal energy density, e_{klm} is permutation symbol ($e_{123} = e_{231} = e_{312} = -e_{132} = -e_{321} = -e_{213} = 1$, and all other $e_{klm} = 0$), m_{kl} is couple stress tensor, φ_k is microrotation vector, j is microinertia, v_k is $\dot{u}_k$, ξ_k is $\dot{\varphi}_k$.

We use rectangular coordinates x_k $(k = 1,2,3)$ or $(x_1 = x$, $x_2 = y$, $x_3 = z)$. Indices following a comma indicate partial differentiation, and a superposed dot indicates the time rate.

Eringen [2] has shown that $0 \le 3\lambda + 2\mu + \vartheta$, $0 \le 2\mu + \vartheta$, $0 \le \vartheta$, $0 \le 3\alpha + \beta + \gamma$, $-\gamma \le \beta \le \gamma$, $0 \le \gamma$.

Upon substituting eqs. (2.4)-(2.5) into (2.1) and (2.2) we obtain

$$(c_1^2+c_3^2)\nabla(\nabla\cdot u)-(c_2^2+c_3^2)\nabla\times(\nabla\times u)+c_3^2\nabla\times\varphi=\ddot{u} \tag{2.6}$$

$$(c_4^2+c_5^2)\nabla(\nabla\cdot\varphi)-c_4^2\nabla\times(\nabla\times\varphi)+\omega_0^2\nabla\times u-2\omega_0^2\varphi=\ddot{\varphi} \tag{2.7}$$

where

$$c_1^2=\frac{\lambda+\mu}{\rho}\ ,\ c_2^2=\frac{\mu}{\rho}\ ,\ c_3^2=\frac{\vartheta}{\rho}$$

$$c_4^2=\frac{\gamma}{\rho j},\ c_5^2=\frac{\alpha+\beta}{\rho}\ ,\ \omega_0^2=\frac{\vartheta}{\rho j} \tag{2.8}$$

We then decompose the vectors u and φ into scalar and vector potentials

$$u=\nabla q+\nabla\times U\ ,\ \nabla\cdot U=0$$

$$\varphi=\nabla\varsigma+\nabla\times\phi\ ,\ \nabla\cdot\phi=0$$

which must verify the equations

$$(c_1^2+c_3^2)\nabla^2 q=\ddot{q}$$

$$(c_4^2+c_5^2)\nabla^2\varsigma-2\omega_0^2\varsigma=\ddot{\varsigma} \tag{2.9}$$

$$(c_2^2+c_3^2)\nabla^2 U+c_3^2\nabla\times\phi=\ddot{U}$$

$$c_4^2\nabla^2\phi-2\omega_0^2\phi+\omega_0^2\nabla\times U=\ddot{\phi}$$

The first two equations (2.9) are uncoupled, while the last two are a coupled system in vectors U and ϕ .

Consider that the plane waves propagating in the positive direction of the unit vector n have the form

$$S=S_0\exp[ik(n\cdot r-vt)] \tag{2.10}$$

where $S=\{q,\varsigma,U,\phi\}$ and $S_0=\{a,b,A,B\}$ with a,b complex constants, A,B complex constant vectors, $k=\dfrac{\omega}{v}$ the wave-number and r the position vector.

By substituting (2.10) into the first two equations (2.9) we obtain

$$v_1^2 = c_1^2 + c_3^2 ,\ v_2^2 = c_4^2 + c_5^2 + 2\frac{\omega_0^2}{k^2} \tag{2.11}$$

where v_1 is the velocity of a longitudinal displacement wave and v_2, the velocity of a longitudinal microrotation wave with its microrotation vector in the direction of the propagation. From the last two equations (2.9) we obtain v_3 and v_4 as roots of the equation

$$(1-2\frac{\omega_0^2}{\omega^2})X^2 - [c_4^2 + c_2^2(1-2\frac{\omega_0^2}{\omega^2}) + c_3^2(1-2\frac{\omega_0^2}{\omega^2})]X + c_4^2(c_2^2 + c_3^2) = 0 \tag{2.12}$$

with $X = v^2$. The remaining two waves are a transverse displacement wave U of velocity v_3 coupled with a transverse microrotation ϕ of speed v_4. These waves exist only for $\omega \geq \sqrt{2}\omega_0$. Below this frequency waves degenerate into sinusoidal vibrations decaying with distance from the source.

3. Determination of the Natural Frequencies

Consider a micropolar plate of thickness $2H$, length a and width b. We locate the Cartesian coordinate system $Oxyz$ at the middle plane denoted by Ω, with the z axis normal to the plane.

We then consider both longitudinal q and ς and transverse U and ϕ waves. Assuming a harmonic time-dependence, we write

$$U(x,t) = U_0(x)e^{i\omega t}\ ,\ \phi(x,t) = \phi_0(x)e^{i\omega t} \tag{3.1}$$

with $U_0(x)$ and $\phi_0(x)$ the unknown functions.

The shear deformation theory proposed by Frederiksen [4] is based on a displacement field in which the displacement in the x and y directions are expanded as cubic functions of the thickness coordinate, and the transverse deflection is assumed to be constant through the thickness.

For a micropolar elastic body we extend this theory by supposing that $S = \{U_0, \phi_0\}$ are cubic functions of the thickness coordinate

$$S_i(x,y,z) = \Psi_i(x,y)z + \Phi_i(x,y)z^2 + \Sigma_i(x,y)z^3 \tag{3.2}$$

with $i = 1,2$. In our notation $S_1 = U_0$, $S_2 = \Phi_0$. Here Ψ, Φ, Σ are unknown expansion functions. Following the Ritz procedure, we assume the solution for Ψ, Φ, Σ to be in the form of finite series with unknown coefficients

$$\Psi_i(x,y) = \sum_{m,n}^{N} X_{imn} w_m(\frac{2x}{a}) w_n(\frac{2y}{b})$$

$$\Phi_i(x,y) = \sum_{m,n}^{N} Y_{imn} w_m(\frac{2x}{a}) w_n(\frac{2y}{b}) \tag{3.3}$$

$$\Sigma_i(x,y) = \sum_{m,n}^{N} Z_{imn} w_m(\frac{2x}{a}) w_n(\frac{2y}{b}),\ i = 1,2,3,4$$

where $\sum_{m.n}^{N} = \sum_{m=0}^{p-1} \sum_{n=0}^{p-1}$ and $w_m(\varepsilon)$, $-1 \le \varepsilon \le 1$ are the assumed functions defined for the nondimensional variable ε. These functions are chosen to satisfy the following requirements: both functions and their first derivatives are continuous; the functions are complete and admissible i.e. satisfy the boundary conditions of the plate. A good set of functions is the one of the degenerated beam functions [4]

$$w_0(\varepsilon) = 1\ ,\ w_1(\varepsilon) = \varepsilon$$

$$w_m(\varepsilon) = cos\ k_I \varepsilon\ ,\ I = \frac{m+2}{2}\ ,\ m = 2,6,10,\ldots$$

$$w_m(\varepsilon) = cosh\ k_J \varepsilon\ ,\ J = \frac{m+1}{2}\ ,\ m = 3,7,11,\ldots \tag{3.4}$$

$$w_m(\varepsilon) = sin\ k_I \varepsilon\ ,\ I = \frac{m+2}{2}\ ,\ m = 4,8,12,\ldots$$

$$w_m(\varepsilon) = sinh\ k_J \varepsilon\ ,\ J = \frac{m+1}{2}\ ,\ m = 5,9,13,\ldots$$

where k_m is the solution of the equation

$$tan\, k_m + (-1)^m\, tanh\, k_m = 0\ ,\ m = 2,3,4,\ldots \tag{3.5}$$

Though the functions (3.4) do not satisfy free edge conditions, it was shown [4] that for problems involving free edges the series composed of these functions converge rapidly towards highly accurate solutions. The arbitrary coefficients X, Y, Z in the series (3.3) are determined from the stationary condition of the functional

$$F = \int_{\Omega} \int_{-H/2}^{H/2} \sum_{i=1}^{4} s_i^2 \tag{3.6}$$

where

$$\begin{aligned} s_1 &= (c_1^2 + c_3^2)\nabla^2 q + \omega^2 q \\ s_2 &= (c_4^2 + c_5^2)\nabla^2 \varsigma - 2\omega_0^2 \varsigma + \omega^2 \varsigma \\ s_3 &= (c_2^2 + c_3^2)\nabla^2 U + c_3^2 \nabla \times \phi + \omega^2 U \\ s_4 &= c_4^2 \nabla^2 \phi - 2\omega_0^2 \phi + \omega_0^2 \nabla \times U + \omega^2 \phi \end{aligned} \tag{3.7}$$

The procedure yields to an eigenvalue problem with ω^2 as the eigenvalue and the unknown coefficients as the eigenvector.

4. Inverse Problem

The elastic material moduli are given by $C = \{\lambda, \mu, \vartheta, \alpha, \beta, \rho\}$. Suppose that $Q^{calc}(C)$, $Q = \{\omega_1 \omega_2 \ldots \omega_N\}$ are theoretical values of the natural frequencies calculated as functions of C, and Q^{mes} the experimental natural frequencies of the plate. To extract the elastic constants from the experimental data, an objective function must be chosen that measures the agreement between theoretical and experimental data

$$\Psi(C) = \frac{\sum_{j=1}^{N} (Q_j^{mes} - Q_j^{calc})^2}{\sum_{j=1}^{N} (Q_j^{mes})^2 + \varepsilon} \tag{4.1}$$

where ε is a small user-specified parameter to avoid division by zero. In our study a genetic algorithm is proposed to determine the initial set of the parameters C. Starting from the parameters thus obtained, more precise inverse analysis is then carried.

For this, we assume that the parameters C are discretized into discrete values so that an arbitrary set of C can be expressed as

$$C = \{\lambda_i, \mu_j, \vartheta_k, \alpha_l, \beta_m, \rho_n\} \tag{4.2}$$

The discretization is performed with steps $\Delta C = \{\Delta\lambda, \Delta\mu, \Delta\vartheta, \Delta\alpha, \Delta\beta, \Delta\rho\}$ counted from the first set of parameters

$$C^0 = \{\lambda_n^0, \mu_n^0, \vartheta_n^0, \alpha_n^0, \beta_n^0, \rho_n^0\} \tag{4.3}$$

We denote with I, J, K, L, M, N the total number of discretized values for each parameter C.

So, the arbitrary set of parameters C given by (4.2) will be characterised by a number

$$S_{ijklmn} = (i-1)JKLMN + (j-1)KLMN + (k-1)LMN + (l-1)MN + (m-1)N + n \qquad (4.4)$$

Such a number indicates a specific set of elastic constants. Each number S represents a gene. We can say that each individual is expressed as a row of the integer numbers. GA is linked to the problem that is to be solved through the fitness function, which measures how well an individual satisfies the real data. From one generation to the next GA usually decreases the objective function of the best model and the average fitness of the population. The starting population (with K individuals) is usually randomly generated. Then, new descendant populations are iteratively created, with the goal of an overall objective function decrease from generation to generation. Each new generation is created from the current one by the main operations: selection, crossover and reproduction, mutation and fluctuation. By selection two individuals of the current population are randomly selected (parent 1 and parent 2) with a probability that is proportional to their fitness.

This ensures that individuals with a good fitness have better chance to advance to the next generation. In the crossover and reproduction operation some crossover sites are chosen randomly and two individuals are reproduced by exchanging some genes between parents. In the new produced individuals, a randomly selected gene is changed with a random generated integer number by the mutation operation. In the fluctuation operation we exchange a discretized value of an unknown parameter in a random direction, by extending the search in the neighbourhood of a current solution. The fitness function is evaluated for each individual that corresponds to the gene representation. The alternation of generations stops when the convergence is detected. Otherwise, the process stops when a maximum number of generations are reached.

5. Numerical Simulations

Computer simulations are carried out for a rectangular plate of $a = 160mm, b = 240mm$ and $H = 38mm$, made from a Gauthier micropar material [4] in which uniformly distributed rigid aluminium shot of diameter $0.7mm$, is cast in an elastic epoxy matrix. The Young elastic modulus, the Poisson ration and the mass density taken from literature are respectively: $E_{al} = 73.1\,GPa$, $\nu_{al} = 0.34$, $\rho_{al} = 2638 Kg/m^3$ for aluminium, and $E_{ep} = 4.21\,GPa$, $\nu_{ep} = 0.31$, $\rho_{ep} = 1060 Kg/m^3$ for epoxy-928 resin. The steps width is given by $\Delta\lambda = \Delta\mu = 10^{-5}\,GPa$, $\Delta\vartheta = 10^{-5}\,MPa$,. $\Delta\alpha = \Delta\beta = \Delta\gamma = 10^{-5}\ GN$, $\Delta\rho = 10^{-5}\,Kg/m^3$. Other parameters of the problem are the spin inertia $j = 6.25\times 10^{-7}\ m^2$, and the starting number of individuals, $K = 20$. The maximum number of generations is 200, the number of multi-point crossovers is 1, and probability of mutation is 0.5.

The first 20 measured natural frequencies (NF) are:

96.54, 115.33, 202.67, 365.65, 432.97, 489.11, 543.3, 774.92, 995.87, 1076.44, 1320.49, 1341.81, 1456.47, 1559.23, 1678, 1805.4, 1988.3, 2290.45, 2654, 2876 (Hz)

A further increase of the number of NF do not give significant improvement in the accuracy.

The values of NF $Q^{calc} = \{\omega_1^{calc}\,\omega_2^{calc}...\omega_N^{calc}\}$ are calculated by using 14×14 functions.

The results given by GA by 307 iterations are:

$\lambda = 7.55 GPa$, $\mu = 6.19 GPa$, $\vartheta = 14.5\,MPa$, $\alpha = 3 GN$, $\beta = 2.8\,GN$, $\gamma = 2.86\,GN$, $\rho = 1160\,Kg/m^3$

Next, by applying a quasi-Newton algorithm the final results are obtained by 34 iterations:

$\lambda = 7.583\,GPa$, $\mu = 6.334\,GPa$, $\vartheta = 14.905\,MPa$, $\alpha = 3.04 GN$, $\beta = 3.12\,GN$, $\gamma = 2.89\,GN$, $\rho = 1189\,Kg/m^3$

6. Conclusions

The model is based on a higher-order shear wave theory which accounts for parabolic distribution of the transversal shear amplitudes of the displacements and microrotations waves through the thickness of the plate.

Due to the wavelengths comparable with the average grain size, the shear displacements waves do not provide sufficient information for a correct solving of the inverse problem.

The identification of elastic properties is fast and simple to perform when both shear displacements and shear microrotation waves are used. The conclusion is that the influence of shear waves on the natural frequencies is very important for micropolar materials. Figures 1-2 give the influence of shear waves (dimensionless quantities $\overline{U}_0$ and $\overline{\Phi}_0$) on the normal vibration eigenmode for a plate of thickness $2H$.

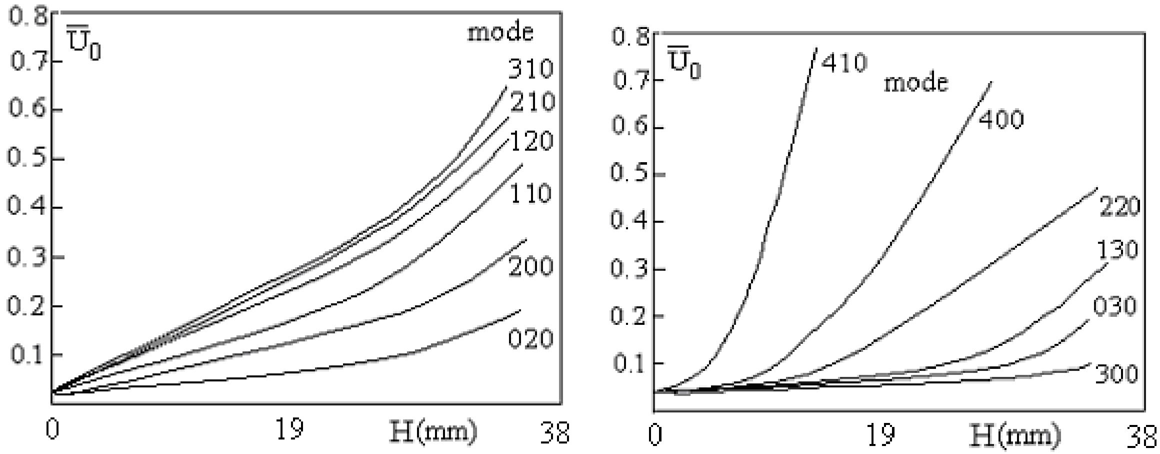

Fig. 1 Influence of dimensionless displacement amplitude $\overline{U}_0$ on the normal vibration eigenmode (xyz) for a micropolar plate of thickness $2H$

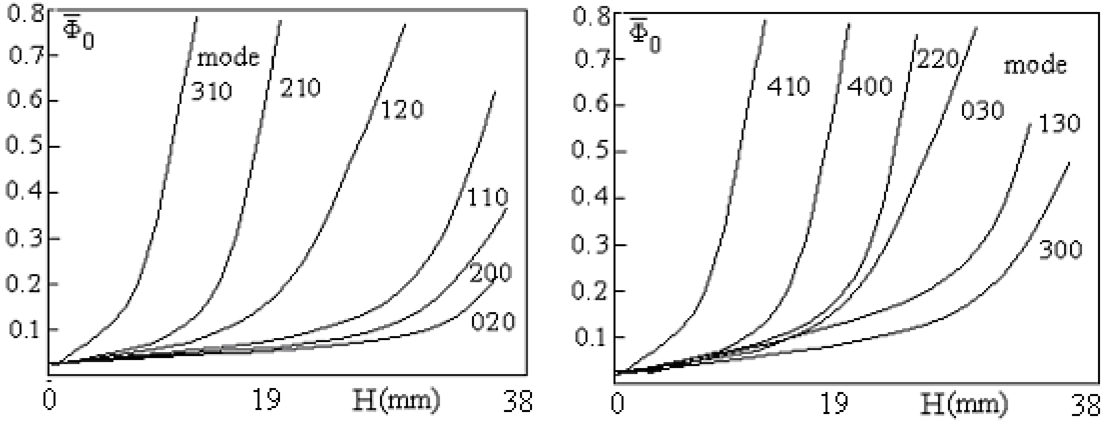

Fig. 2 Influence of dimensionless microrotation amplitude $\overline{\Phi}_0$ *on the normal vibration eigenmode (xyz) for a micropolar plate of thickness* $2H$

The analysis has also shown that the presence of micropoles (aluminium shot) introduces significant changes in the natural frequencies configuration. A micropole appears as a disturbance in the normal vibration eigenmode pattern.

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